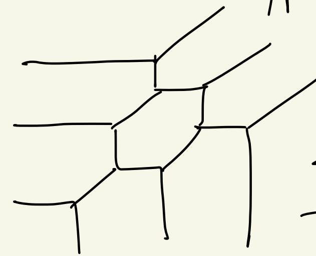
Realisability of tropical varieties

Week 1 Tropical polynomials + hypersurfaces.

Thm (Mikhalkin)
Tropical curves in $\mathbb{R}^2$ are precisely the weighted rational P.L. graphs in $\mathbb{R}^2$ satisfying the balancing condition.



Trop.

$$f = \bigoplus a_\alpha \odot x^\alpha \longleftarrow F = \sum A_\alpha X^\alpha$$

Week 2 Tropicalisation of varieties / $\mathbb{K} = k\{\{t\}\}$.

Thm (Kapranov) $f \in \mathbb{K}[\underline{x}]$.

$$V_{Trop}(\mathrm{Trop}(f)) = \mathrm{Trop}(V(f))$$

Mikhalkin
Kapranov $\Longrightarrow$ $\forall$ $\mathbb{Q}$-tropical hypersurface $\subseteq \mathbb{R}^n$ arises as the tropicalisation of a hypersurface / $\mathbb{K}$.

Tropicalising min vs. max.

<u>Recall</u>. $\mathbb{K} = \mathbb{C}\{\{t\}\}$ and $V \subseteq (\mathbb{K}^*)^n$ algebraic variety

$$\mathrm{Trop}(V) := \{(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) \mid (x_1, \ldots, x_n) \in V\}$$

(together with weights)

$$\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}.$$

$$\Updownarrow$$

$$-\mathrm{val}(a+b) \leq \max\{-\mathrm{val}(a), -\mathrm{val}(b)\}.$$

To use "max" with val define $\mathrm{Trop}(V) := \overline{\{(-\mathrm{val}(x_1), \ldots, -\mathrm{val}(x_n))\}}$

Properties of Tropical Varieties

$V \subseteq (\mathbb{K}^*)^n$ is a variety defined over $\mathbb{K} = \mathbb{C}\{\{t\}\}$.

1) $\mathrm{Trop}(V) \subseteq \mathbb{R}^n$ is a rational polyhedral complex.

target space $T_\sigma \supset L_\sigma \leftarrow$ rk dim σ $\mathbb{Z}$-lattice $\qquad \mathrm{Trop}(V) = \bigcup_{\sigma \text{ polyhedra}} \sigma$

2) $\dim V = \dim \mathrm{Trop}(V)$.

3) $\deg V = \deg \mathrm{Trop}(V)$. $\qquad \deg V := \deg \overline{V} \qquad \overline{V} \subseteq \mathbb{KP}^n$.

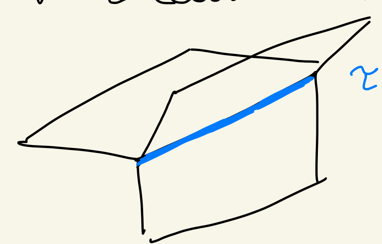
4) $\mathrm{Trop}(V)$ is equipped with $\mathbb{N}_{>0}$ weights on top dim faces Facets.

5) $\mathrm{Trop}(V)$ with $\{w_\sigma\}_{\sigma \in \text{Facet}}$ is balanced: $\forall\ \tau$ codim 1 face

$$\sum_{\substack{\sigma \in \text{Facet} \\ \tau \subset \sigma}} w_\sigma v_{\sigma/\tau} \in \langle \tau \rangle$$

where $v_{\sigma/\tau}$ s.t.

$L_\sigma = \langle L_\tau, v_{\sigma/\tau} \rangle_{\mathbb{Z}}$.



Tropical Varieties in $\mathbb{R}^n$

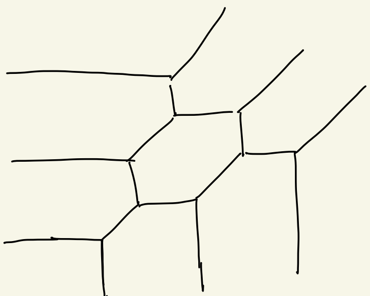
Definition. A *tropical variety* in $\mathbb{R}^n$ is a rational polyhedral complex together with a weight function $w: \text{Facets } X \to \mathbb{N}_{>0}$ satisfying the balancing condition.

Examples hypersurfaces, tropicalisations, matroid fans, ...

The Realisability Question Given a tropical variety $X \subseteq \mathbb{R}^n$ with weights $w: \text{Facets } X \to \mathbb{N}_{>0}$ does there exist a variety $V/\mathbb{K}$ such that $\text{Trop}(V) = \text{Trop}(X)$. ?

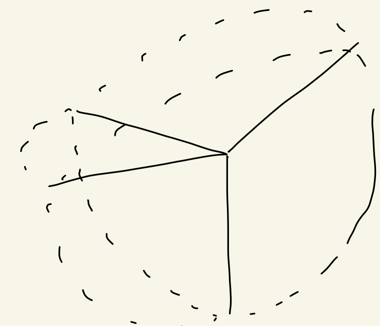
First Example (Mikhalkin)(Speyer 2014) $b_1(C)$

There exists tropical cubic curves of genus 1 in space which are not contained in a tropical plane.



$\mathrm{Trop}(\mathcal{C}) = C \subseteq \mathbb{R}^2$

$\tilde{C} \subseteq \mathbb{R}^3$



$\mathrm{Trop}(P) = P \subseteq \mathbb{R}$

By Castelnuovo's bound $\tilde{C} \subseteq \mathbb{R}^3$ can not be realisable.

Exercise

$\exists$ trop curves of degree $3 \subseteq \mathbb{R}^3$ with genus $= 2$! (Bertrand-Brugallé-López de Medrano)

Realisability & Enumerative Geometry.

$GW_{d,g}$ = # of irred genus g degree d curves through $P_1 \dots P_{3d+g-1}$ general points in $\mathbb{CP}^2$.

Mikhalkin's Theorem In $\mathbb{P}^2$ (2000).

computed by Kontsevich 1994

$$GW_{d,g} = GW^{Trop}_{d,g} = \sum_{\substack{C \supset \{p_1, \dots, p_{3d+g-1}\} \\ C \text{ irr trop curve} \\ \deg(C) = d \quad g(C) = g}} m(C)$$

$$m(C) = \prod_{v \text{ 3-valent vertex}} \operatorname{mult}(v)$$

$$\operatorname{mult} v = w(e_1) w(e_2) \left| \det(v_1, v_2) \right|$$

Lines in cubic surfaces

Cayley-Salmon Thm (1849) A non-singular cubic surface S over an algebraically closed field contains exactly 27 lines

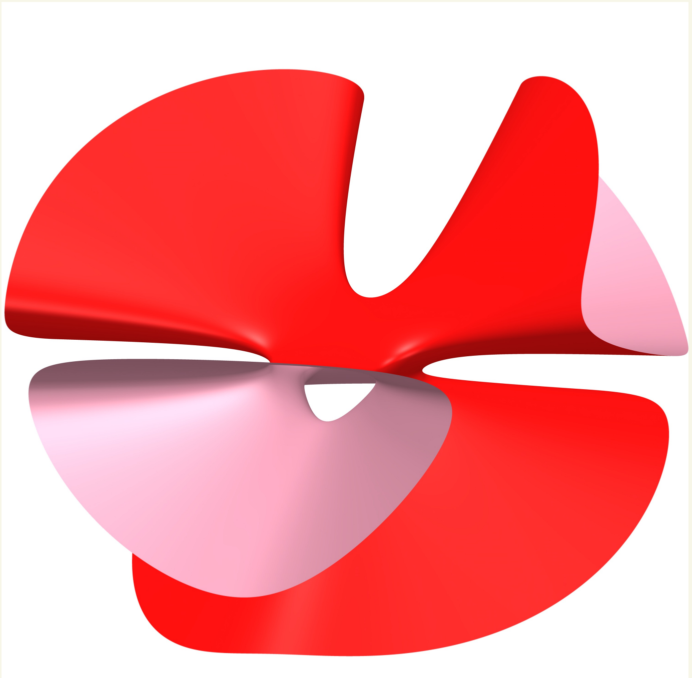
Schläfli (1858) A non-singular cubic surface over $\mathbb{R}$ contains either 3, 7, 15, or 27 lines

Vigeland (2007) There $\exists$ non-singular tropical cubic surfaces containing more than 27 lines.

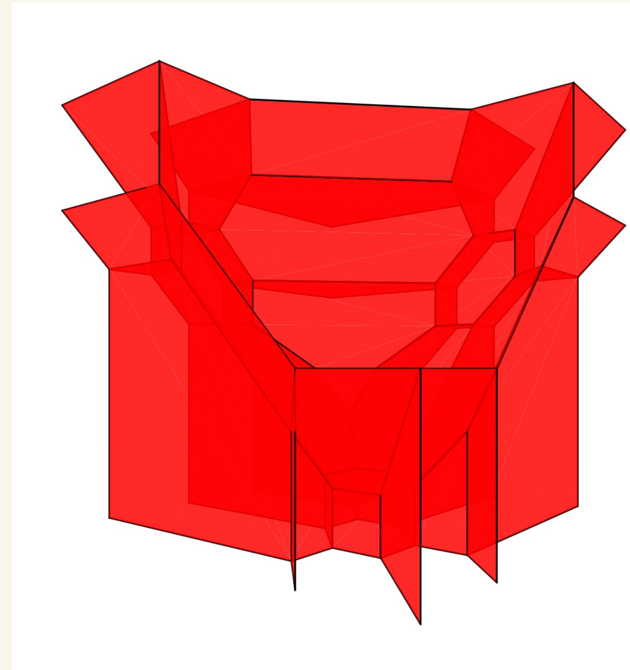
Complete classification finished by Panizzut Vigeland 2021

Over 14 million triangulations of size 3 simplex. (Jordan, Joswig, Kastner 2018)

"Clebsch" ℝ-cubic surface



Tropical cubic surface



Realising Curves in Surfaces

Let C be a trop. curve contained in a trop surface X. The pair (X, C) is <u>realisable</u> if

$\exists \ \mathcal{C} \subset \mathcal{X} / \mathbb{K}$ s.t. $\mathrm{Trop}(\mathcal{C}) = C \quad \mathrm{Trop}(\mathcal{X}) = X$.

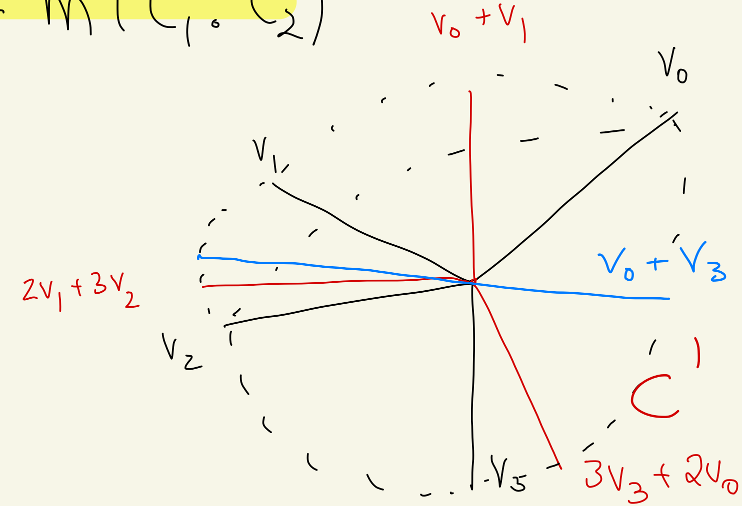
Open Question Are there $\mathcal{X}_1, \mathcal{X}_2$ realising a cubic surface X s.t. the tropicalisations of the two sets of 27 lines are distinct?

Theorem (Global to Local)
If $C \subset X$ is a realisable pair $C_p \subset X_p$ is a realisable pair for every $p \in C$.

Local Obstructions from intersection Theory

Thm (Brugallé - S. 2011) Let $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{P}$ be two curves contained in a plane defined over $\mathbb{K}$ s.t. $\mathrm{Trop}(\mathcal{C}_1)$, $\mathrm{Trop}(\mathcal{C}_2)$ are fans intersecting in a point then

$$m(\mathrm{Trop}(\mathcal{C}_1).\mathrm{Trop}(\mathcal{C}_2)) = m(\mathcal{C}_1 . \mathcal{C}_2)$$

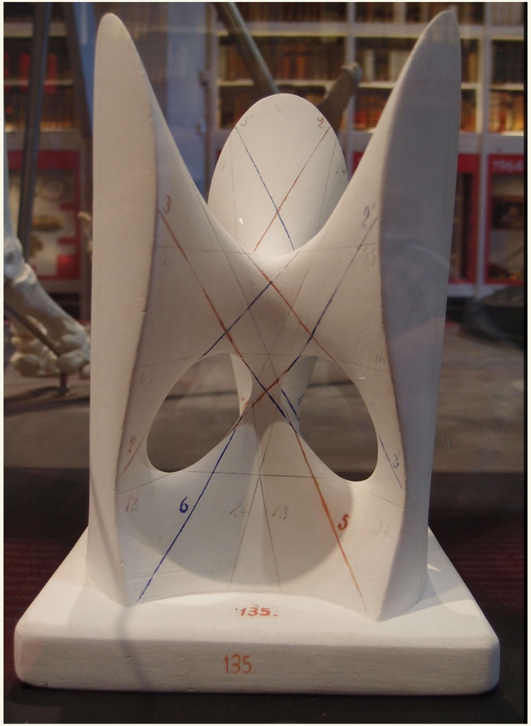


Corollary. Let $C \subset \mathrm{Trop}(\mathcal{P})$ be a tropical curve. If

$C . \mathrm{Trop}(\mathcal{C}') < 0$ for some $\mathcal{C}' \subset \mathcal{P}$ then C is not realizable in $\mathcal{P}$!

Rationality of lines in cubic surfaces.

Fact. If $S/_{\mathbb{R}\{\{t\}\}}$ is a cubic surface s.t. Trop(S) is non-singular then for t sufficiently small all lines on S_t are real.



Future Thm Let S be a cubic surface defined over a valued field s.t. trop(S) is non-singular. Then all lines in S are rational.

Eg. $S/_{\mathbb{Q}_p}$ val = p-adic valuation

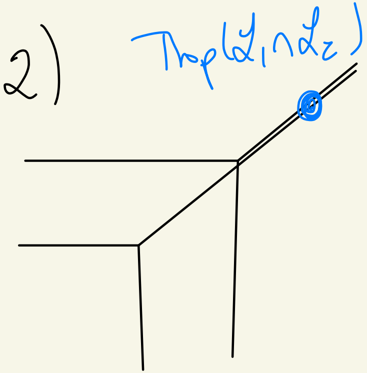
Summary

- Tropical varieties are weighted rational polyhedral complexes satisfying the balancing condition.
- Realisability Question asks: when is a trop variety = Trop(V) for some V / $\mathbb{K}$ valued field?
- All tropical hypersurfaces in $\mathbb{R}^n$ are realisable
  $\Longrightarrow$ tropical count of GW invariants + generalisations.
- "Easy" to construct examples of non-realisable trop varieties
- Bug or Feature? Next time: Matroids + tropical geometry
  Mannino.

Exercises.

1) a) Show there is exactly 1 tropical line through points $p_1, p_2 \in \mathbb{R}^2$ if the points are contained on a line with rational slope.

b) Experiment with drawing tropical conics through 5 points

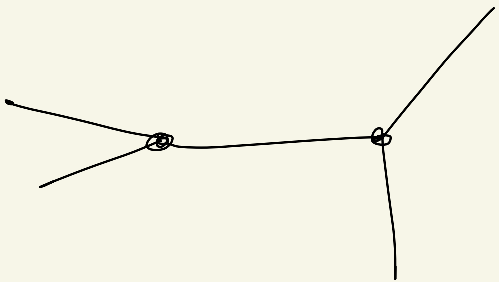
2) 

Let L_1, L_2 be the tropical lines with vertices $(0,0)$ and $(1,1)$, respectively. Show that for any $b \in \mathbb{R}$ with $b \geq$ there exist realizations $\mathcal{L}_1, \mathcal{L}_2$ s.t.

$$\mathrm{Trop}(\mathcal{L}_1 \cap \mathcal{L}_2) = (b, b).$$

3) Show that the polyhedral complex with vertices:

$V_1 = (0,0,0)$ $V_2 = (1,1,0)$ and



edges:

$E_0 = \{(a,a,0) \mid 0 \le a \le 1\}$

$E_1 = \{(a,0,0) \mid a \leqslant 0\}$

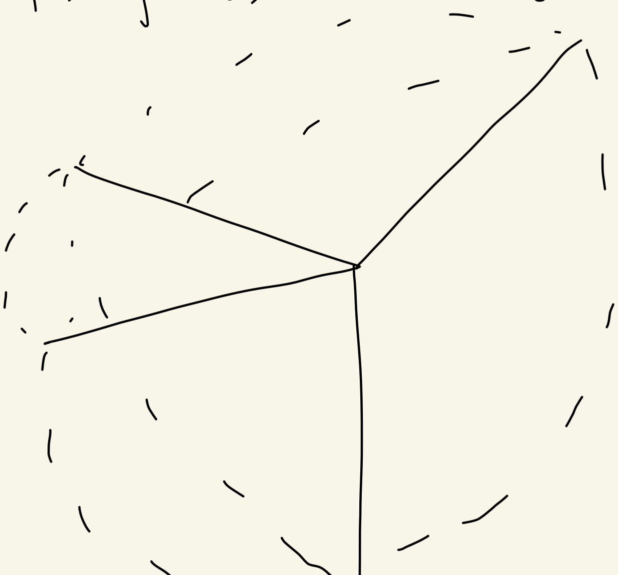
$E_2 = \{(0,a,0) \mid a \leqslant 0\}$

$E_3 = \{(1,1,a) \mid a \leqslant 0\}$

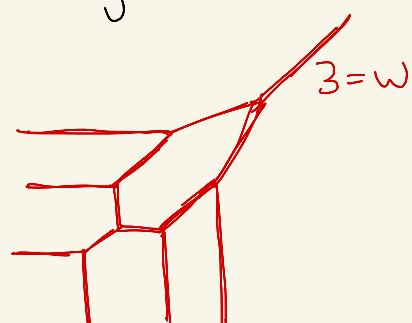
$E_4 = \{(a+1,a+1,a) \mid a \geqslant 0\}$.

is a tropical variety and realisable by a $\mathcal{L} \cap (\mathbb{K}^*)^3$ where $\mathcal{L} \subseteq \mathbb{KP}^3$ is a line and $\mathbb{K} = \mathbb{C}\{\{t\}\}$.

4) The tropical plane $P \subseteq \mathbb{R}^3$ is a fan with 4 rays in directions $v_i = -e_i$ $i = 1,..,3$ and $v_0 = (1,1,1)$ and 6 two dimensional cones $\langle v_i, v_j \rangle_{\geq 0}$ $i \neq j$.

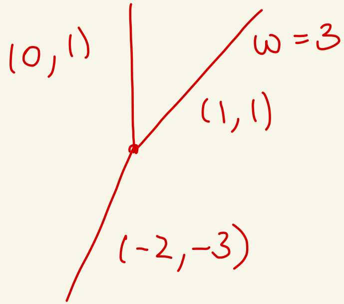


Construct a tropical curve $C \subset P$ s.t. $b_1(C) = 2$ and C projects to a degree 3 curve. where forgetting e_3.

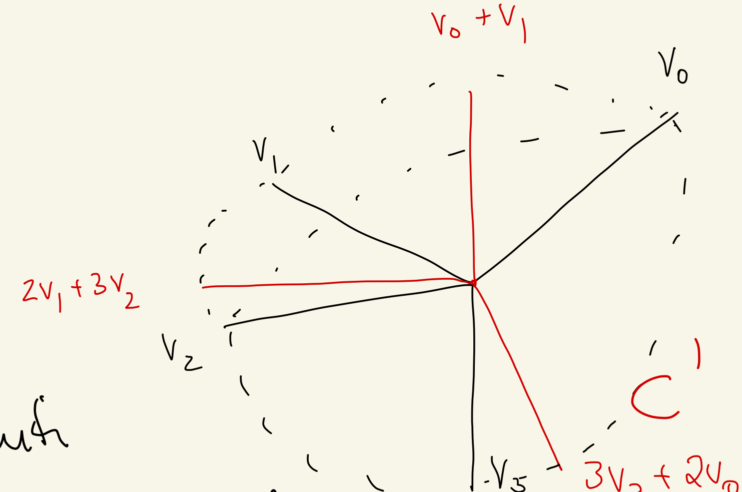


5)

a) Consider the tropical curve C in $\mathbb{R}^2$ in red. Argue using its Newton polygon that any realisation of C would have a cusp at $(0,0)$.



b) Let $\mathcal{P} \subseteq (\mathbb{K}^*)^3$ be a plane defined by $z_1+z_2+z_3=0$. Show there is no $\mathcal{C} \subset \mathcal{P}$ of degree 3 tropicalising to C'.



<u>Hint</u>: Consider the projections $(\mathbb{K}^*)^3$ with kernel $\langle e_2 \rangle$ and $\langle e_3 \rangle$ to show a realisation would have 2 cusps.